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Numerical Solution of Eigenvalue Problems Using Spectral Techniques

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Two algorithms based on spectral Chebyshev and pseudospectral Chebyshev methods are presented for solving difficult eigenvalue problems that are valid over connected domains coupled through interfacial conditions. To demonstrate the applicability of these methods, we have examined the eigenvalue problems that describe the linear stability of two superposed Newtonian and inelastic power law fluids in plane Poiseuille flow for a selected range of parameters. Both algorithms provide accurate results and the pseudospectral code appears to be more efficient in handling linear stability problems.

1. INTRODUCTION

The compound matrix method was formulated by Ng and Reid [1, 2] to deal with boundary value and eigenvalue of stiff differential operators with separated boundary conditions. More recently, Yiantsios and Higgins [3] extended the compound matrix method to equation sets valid over connected domains coupled through interfacial conditions and examined the linear stability of two superposed Newtonian fluids in plane Poiseuille flow. In general it has been shown by Ng and Reid [1, 2] that the compound matrix method is superior to the shooting method for solutions of stiff eigenvalue and boundary value problems. However, similar to shooting techniques it can only track a single eigenvalue. Moreover, since an iterative technique is used to calculate the eigenvalue a good initial guess for the eigenvalue is required. When applying this technique to eigenvalue problems which describe the stability of various systems a knowledge of stability of other modes is also required. Hence, as suggested by Yiantsios and Higgins [3] it is advantageous to use a method that calculates all the eigenvalues for the described problem (i.e., finite elements) and then refine the calculations for a desired mode.

The spectral Chebyshev tau method first developed by Orzsag [4] to investigate the linear stability of a single fluid in plane Poiseuille flow can also be used to solve stiff boundary value and eigenvalue problems. This method approximates discrete eigenvalues belonging to c^{∞} eigenfunctions with infinite order accuracy. Unlike the compound matrix method, with this technique all eigenvalues of the spectrum can be obtained accurately without the requirement of a good initial guess.

In this work we have extended the spectral Chebyshev tau method to equation sets valid over connected domains, coupled through interfacial conditions. Since the spectral Chebyshev tau method is only valid for differential equations with polynomial coefficients, a pseudospectral Chebyshev method is also developed to handle differential equations with arbitrary coefficients. To demonstrate the applicability of these techniques we have examined the eigenvalue problems which describe the linear stability of two superposed Newtonian and inelastic power law fluids in plane Poiseuille flow.

2. SPECTRAL TECHNIQUES

2.1. *The Spectral Chebyshev Tau Method*

In linear stability analysis of multiphase flows, the evolution equation can be written as

$$M_i(\phi_i)=0, \qquad i=1, 2, ..., I, \tag{1}$$

where ϕ_i is the solution of ith layer and M_i is an operator for ith layer, which contains all the spatial derivatives of ϕ_i. The linear stability of such flows is described by (1) and the rigid and interfacial boundary conditions.

The discretized solutions to (1) are represented by

$$\phi_i^N(y)=\sum_{n=0}^{N} a_n^{(i)}\,\psi_n(y), \qquad n=0, 1, ..., N;\ i=1, 2, ..., I, \tag{2}$$

where ψ_n's are the trial functions, and the $a_n^{(i)}$'s are the expansion coefficients. It has been shown by Orzsag [4] that Chebyshev polynomials provide excellent trial func-

0021-9991/92 $5.00

tions because of their rapid convergence. Hence, we use these polynomials as our trial functions. In general, the trial functions do not satisfy the governing equations and boundary conditions. Thus, it is necessary to have weighted residual conditions (WR). The test functions chosen for the Chebyshev trial functions are

$$\chi_n(y) = \frac{2}{c_n \pi \sqrt{1-y^2}} T_n(y), \qquad n = 0, 1, ..., N, \tag{3}$$

where

$$c_n = 0, \qquad n < 0$$
$$c_n = 2, \qquad n = 0$$
$$c_n = 1, \qquad n > 0$$

and the weighted residual conditions are given by

$$\int_{-1}^{1} M_i(\phi_i^N)\, \chi_n(y)\, dy = 0,$$
$$i = 1, 2, ..., I; n = 0, 1, ..., N, \tag{4}$$

upon expansion of equations resulting from this expression and boundary conditions in terms of Chebyshev polynomials and application of tau method an algebraic eigenvalue problem of the form $(A_n \lambda^n + \cdots + A_1 \lambda + A_0) X = 0$ is obtained. Upon transformation of this expression a standard generalized eigenvalue problem of the form $(A - B\lambda) X = 0$ results, the eigenvalues of which can be determined by use of a generalized matrix eigenvalue solver such as the QR algorithm.

2.2. *The Pseudospectral Chebyshev Method*

It has been shown by Orszag [5] and Kreiss and Oliger [6] that the pseudospectral method (i.e., in this paper pseudospectral method refers to the formulation in the expansion coefficients) can be applied efficiently in the physical space in contrast to the spectral Chebyshev tau method which must be carried out in the transformed space. This fact enables us to use the pseudospectral method efficiently for differential equations with arbitrary coefficients.

In the pseudospectral method the test functions are shifted Dirac delta-functions $\delta(y\text{-}y_j)$, where y_j's are collocation points in the interval $(-1, 1)$. Upon substitution of shifted Dirac delta-functions into (4), the weighted residual condition for the pseudospectral method is obtained:

$$\int_{-1}^{1} M_i(\phi_i^N)\, \delta(y - y_j)\, dy = 0, \qquad i = 1, 2, ..., I. \tag{5}$$

This formulation requires that (1) be satisfied exactly at collocation points. Hence,

$$M_i(\phi_i^N)|_{y = y_j} = 0, \qquad i = 1, 2, ..., I. \tag{6}$$

Similar to the spectral Chebyshev tau method, the equations resulting from (6) and the boundary conditions are expanded in terms of Chebyshev polynomials and an algebraic eigenvalue problem is obtained.

In the following two examples, the detailed procedure for the formulation of standard generalized eigenvalue problems of equation sets valid over connected domains coupled through interfacial condition is outlined.

3. TWO SUPERPOSED NEWTONIAN FLUIDS

We consider two Newtonian fluids flowing steadily in two distinct layers between two parallel plates. Since detailed derivations of governing equations and boundary conditions were given by Yih [7], only a brief account is presented in the following.

The unperturbed velocity profiles nondimensionalized with respect to the interfacial velocity U_0 are

$$U_k = 1 + e_k y + f_k y^2, \qquad k = 1, 2, \tag{7}$$

where

$$e_1 = \frac{m_2 - \varepsilon^2}{\varepsilon^2 + \varepsilon}, \qquad f_1 = \frac{-(m_2 + \varepsilon)}{(\varepsilon^2 + \varepsilon)},$$
$$e_2 = e_1 / m_2, \qquad f_2 = f_1 / m_2,$$

in which $m_k = \mu_k / \mu_1$ and $\varepsilon = d_2 / d_1$ are the viscosity and depth ratios, respectively. The subscripts 1 and 2 correspond to the upper and lower layers and all the physical and geometric parameter ratios are defined in terms of the lower layer properties to that of the upper layer.

To study the interfacial stability of this system, we introduced two-dimensional infinitesimal disturbances into equations of motion and continuity, and then linearized these equations with respect to the perturbation quantities. As is customary in stability analyses, we assume that all perturbation quantities have an exponential time dependence and periodic spatial dependence of the form $\exp i\alpha(x - ct)$, where α is the dimensionless wave number of the disturbance in the flow direction x, c is the dimensionless complex wave speed, and t is the dimensionless time. Upon substitution of these variables into equations of motion and continuity and elimination of the pressure terms the stability governing equations of two superposed Newtonian fluids are obtained. These equations are

$$\frac{i\alpha \operatorname{Re} r_k}{m_k} \{(U_k - c)(\phi_k'' - \alpha^2 \phi_k) - U_k'' \phi_k\}$$
$$= \phi_k'''' - 2\alpha^2 \phi_k'' + \alpha^4 \phi_k, \qquad k = 1, 2. \tag{8}$$

The above equations, along with no-slip boundary conditions and the continuity of velocities and stresses at the interface shown below, govern the stability of two immiscible Newtonian fluids,

$$\begin{aligned}
&\phi_1=\phi_1'=0 && \text{at } y=1\\
&\phi_2=\phi_2'=0 && \text{at } y=-\varepsilon\\
&\phi_1=\phi_2 && \text{at } y=0\\
&\phi_1'-\phi_2'=\phi_1(e_2-e_1)/(c-1) && \text{at } y=0\\
&\phi_1''+\alpha^2\phi_1=m_2(\phi_2''+\alpha^2\phi_2) && \text{at } y=0\\
&m_2(\phi_2'''-3\alpha^2\phi_2')-(\phi_1'''-3\alpha^2\phi_1')\\
&\quad+i\alpha\,\mathrm{Re}\,r_2[(c-1)\,\phi_2'+e_2\phi_2]\\
&\quad-i\alpha\,\mathrm{Re}[(c-1)\,\phi_1'+e_1\phi_1]\\
&\quad=i\alpha\,\mathrm{Re}(F+\alpha^2S)\,\phi_1/(c-1) && \text{at } y=0,
\end{aligned}\tag{9}$$

where $r_k=\rho_k/\rho_1$ is the density ratio, $\mathrm{Re}=\rho_1U_0d_1/\mu_1$ is Reynolds number, $F=(r_2-1)\,gd_1/U_0^2$, and $S=\sigma/\rho_1d_1U_0^2$ are dimensionless groups expressing the effects of gravity g and interfacial tension σ. The above governing equations along with eight boundary conditions can be solved by use of the spectral Chebyshev method mentioned in the last section.

Due to arbitrary depth ratios being considered, the governing equations and boundary conditions are needed to be linearly transformed to the range of $-1\leqslant y\leqslant 1$. We seek approximate solutions of the form

$$\begin{aligned}
\phi_1&=\sum_{n=0}^{N}a_n^{(1)}T_n(y)\\
\phi_2&=\sum_{n=0}^{N}a_n^{(2)}T_n(y)
\end{aligned}\tag{10}$$

to (8). Equations for the expansion coefficient $a_n^{(1)}$ and $a_n^{(2)}$ are found by formally substituting (10) into (8). Upon use of the WR conditions (4), and equating the coefficients of the various $T_n(y)$ to zero, we obtain the equations

$$\begin{aligned}
&\frac{2}{3(1+\varepsilon)^4}\sum_{\substack{p=n+4\\p\equiv n(\mathrm{mod}\,2)}}^{N}[p^3(p^2-4)^2-3n^2p^5\\
&\quad+3n^4p^3-n^2(n^2-4)^2p]\,a_p^{(k)}\\
&\quad-\sum_{\substack{p=n+2\\p\equiv n(\mathrm{mod}\,2)}}^{N}\left\{\left[\frac{4}{(1+\varepsilon)^2}\left(2\alpha^2+\frac{ir_k\alpha\,\mathrm{Re}}{m_k}(1-c)\right)\right.\right.\\
&\quad+\frac{i\alpha\,\mathrm{Re}\,r_k}{m_k}\left(2e_k\frac{1-\varepsilon}{(1+\varepsilon)^2}+f_k\frac{(1-\varepsilon)^2}{(1+\varepsilon)^2}\right)\\
&\quad\left.+\frac{i\alpha\,\mathrm{Re}\,r_kf_k}{4m_k}(c_n+c_{n-1})\right]\\
&\quad\times p(p^2-n^2)+\frac{ir_k\alpha\,\mathrm{Re}f_k}{4m_k}d_{n-2}\\
&\quad\times p\left(p^2-(n-2)^2\right)+\frac{i\alpha r_k\,\mathrm{Re}f_k}{4m_k}\\
&\quad\left.\times c_np(p^2-(n+2)^2)\right\}a_p^{(k)}-\sum_{\substack{p=n+1\\p+n\equiv 1(\mathrm{mod}\,2)}}^{N}\frac{i\alpha\,\mathrm{Re}\,r_k}{m_k}\\
&\quad\times\left(\frac{e_k}{1+\varepsilon}+f_k\frac{1-\varepsilon}{1+\varepsilon}\right)[d_{n-1}p(p^2-(n-1)^2)\\
&\quad+c_np(p^2-(n+1)^2)]\,a_p^{(k)}\\
&\quad-d_{n-2}\frac{i\alpha\,\mathrm{Re}\,r_kf_k}{m_k}n(n-1)\,a_n^{(k)}\\
&\quad+\left\{\alpha^4+2f_k\frac{i\alpha\,\mathrm{Re}\,r_k}{m_k}+\frac{i\alpha^3\,\mathrm{Re}\,r_k}{m_k}\right.\\
&\quad\left.\times\left[(1-c)+\frac{e_k(1-\varepsilon)}{2}+f_k\frac{(1-\varepsilon)^2}{4}\right]\right\}c_na_n^{(k)}\\
&\quad+\frac{i\alpha^3\,\mathrm{Re}f_kr_k}{16m_k}(1+\varepsilon)^2\\
&\quad\times\left[a_{n-2}^{(k)}c_{n-2}+c_na_n^{(k)}(c_n+c_{n-1})+a_{n+2}^{(k)}c_n\right]\\
&\quad+\frac{i\alpha^3\,\mathrm{Re}\,r_k}{4m_k}[e_k(1+\varepsilon)+f_k(1-\varepsilon^2)]\\
&\quad\times(a_{n-1}^{(k)}c_{n-1}+a_{n+1}^{(k)}c_n)=0,
\end{aligned}\tag{11}$$

where $c_n=0$ if $n<0$, $c_0=2$, $c_n=1$ if $n>0$, and $d_n=0$ if $n<0$, $d_n=1$ if $n\geqslant 0$.

Boundary conditions expanded in terms of Chebyshev polynomials are

$$\begin{aligned}
&\sum_{n=0}^{N}a_n^{(1)}=0\\
&\sum_{n=0}^{N}n^2a_n^{(1)}=0\\
&\sum_{n=1}^{N}(-1)^{n-1}n^2a_n^{(2)}=0\\
&\sum_{n=0}^{N}(-1)^na_n^{(2)}=0\\
&\sum_{n=0}^{N}[a_n^{(1)}-a_n^{(2)}]\,T_n(\varepsilon_1)=0\\
&\sum_{n=0}^{N}\left\{-\frac{2}{c_n}\sum_{\substack{p=n+1\\p+n\equiv 1(\mathrm{mod}\,2)}}^{N}pa_p^{(1)}+\frac{2}{c_n}\sum_{\substack{p=n+1\\p+n\equiv 1(\mathrm{mod}\,2)}}^{N}pa_p^{(2)}\right.\\
&\quad+\frac{1+\varepsilon}{2}e_1a_n^{(1)}-\frac{1+\varepsilon}{2}e_2a_n^{(2)}\\
&\quad\left.-\frac{2c}{c_n}\sum_{\substack{p=n+1\\p+n\equiv 1(\mathrm{mod}2)}}^{N}pa_p^{(2)}+\frac{2c}{c_n}\sum_{\substack{p=n+1\\p+n\equiv 1(\mathrm{mod}2)}}^{N}pa_p^{(1)}\right\}T_n(\varepsilon_1)=0
\end{aligned}$$

$$
\begin{aligned}
&\sum_{n=0}^{N}\left\{\frac{4}{(1+\varepsilon)^2 c_n}\left[\sum_{\substack{p=n+2\\ p\equiv n(\mathrm{mod}\ 2)}}^{N} p(p^2-n^2)\,a_p^{(1)} - m_2 \right.\right.\\
&\qquad \left.\times \sum_{\substack{p=n+2\\ p\equiv n(\mathrm{mod}\ 2)}}^{N} p(p^2-n^2)\,a_p^{(2)}\right]\\
&\qquad \left. -\alpha^2[m_2 a_n^{(2)} - a_n^{(1)}]\right\} T_n(\varepsilon_1) = 0\\
&\sum_{n=0}^{N}\left\{-i\alpha\,\mathrm{Re}\left[\frac{4c}{(1+\varepsilon)\,c_n}\sum_{\substack{p=n+1\\ p+n\equiv 1(\mathrm{mod}\ 2)}}^{N} p a_p^{(1)}\right.\right.\\
&\qquad \left. -\frac{4}{(1+\varepsilon)\,c_n}\sum_{\substack{p=n+1\\ p+n\equiv 1(\mathrm{mod}\ 2)}}^{N} p a_p^{(1)} + e_1 a_n^{(1)}\right]\\
&\qquad +\frac{12\alpha^2}{(1+\varepsilon)\,c_n}\sum_{\substack{p=n+1\\ p+n\equiv 1(\mathrm{mod}\ 2)}}^{N} p a_p^{(1)} - \frac{64}{c_n(1+\varepsilon)^3}\\
&\qquad \times[(n+1)(n+2)(n+3)\,a_{n+3}^{(1)} + 3(n+2)\\
&\qquad \times(n+3)(n+5)\,a_{n+5}^{(1)} + 6(n+3)\\
&\qquad \left.\times(n+4)(n+7)\,a_{n+7}^{(1)} + \cdots]\right\} T_n(\varepsilon_1)\\
&\qquad +\sum_{n=0}^{N}\left\{i\alpha\,\mathrm{Re}\,r_2\left[\frac{4c}{(1+\varepsilon)\,c_n}\sum_{\substack{p=n+1\\ p+n\equiv 1(\mathrm{mod}\ 2)}}^{N} p a_p^{(2)}\right.\right.\\
&\qquad \left. -\frac{4}{(1+\varepsilon)\,c_n}\sum_{\substack{p=n+1\\ p+n\equiv 1(\mathrm{mod}\ 2)}}^{N} p a_p^{(2)} + e_2 a_n^{(2)}\right]\\
&\qquad -\frac{12\alpha^2 m_2}{(1+\varepsilon)\,c_n}\sum_{\substack{p=n+1\\ p+n\equiv 1(\mathrm{mod}\ 2)}}^{N} p a_p^{(2)}\\
&\qquad +\frac{64 m_2}{(1+\varepsilon)^3 c_n}[(n+1)(n+2)(n+3)\,a_{n+3}^{(2)}\\
&\qquad +3(n+2)(n+3)(n+5)\,a_{n+5}^{(2)}\\
&\qquad \left. +6(n+3)(n+4)(n+7)\,a_{n+7}^{(2)} + \cdots]\right\}\\
&\qquad \times T_n(\varepsilon_1) - \sum_{n=0}^{N}\frac{i\alpha\,\mathrm{Re}(F+\alpha^2 S)}{(c-1)}\,a_n^{(1)} T_n(\varepsilon_1) = 0, \qquad (12)
\end{aligned}
$$

where $\varepsilon_1 = (\varepsilon-1)/(\varepsilon+1)$.

TABLE I

Interfacial Mode Eigenvalues for Two Superposed Newtonian Fluids

Viscosity ratio m_2	c (Ref. [7])	c (Spectral method)
100	$2.71932287 + 2.05299 \times 10^{-5}i$	$2.71932 + 2.05300 \times 10^{-5}i$
60	$2.56766494 + 8.26910 \times 10^{-6}i$	$2.56766 + 8.26909 \times 10^{-6}i$
20	$2.06020558 + 1.58950 \times 10^{-6}i$	$2.06021 + 1.58952 \times 10^{-6}i$
10	$1.67219917 + 1.24810 \times 10^{-6}i$	$1.67220 + 1.24810 \times 10^{-6}i$
5	$1.33333333 + 7.53400 \times 10^{-7}i$	$1.33333 + 7.53439 \times 10^{-7}i$

Note. $\alpha = 1.0 \times 10^{-5}$, Re $= 10.0$, $r_2 = 1$, $F = 0$, $S = 0$, $\varepsilon = 1.0$.

TABLE II

Interfacial Mode Eigenvalues for Two Superposed Newtonian Fluids

Viscosity ratio m_2	c (Ref. [7])	c (Spectral method)
100	$2.71932287 + 2.05299 \times 10^{-2}i$	$2.71925 + 2.05258 \times 10^{-2}i$
60	$2.56766490 + 8.26910 \times 10^{-3}i$	$2.56747 + 8.26842 \times 10^{-3}i$
20	$2.06020558 + 1.58950 \times 10^{-3}i$	$2.06008 + 1.58908 \times 10^{-3}i$
10	$1.67219917 + 1.24810 \times 10^{-3}i$	$1.67213 + 1.24773 \times 10^{-3}i$
5	$1.33333333 + 7.53400 \times 10^{-4}i$	$1.33333 + 7.5291 \times 10^{-4}i$

Note. $\alpha = 1.0 \times 10^{-2}$, Re $= 10.0$, $r_2 = 1$, $F = 0$, $S = 0$, $\varepsilon = 1.0$.

In order to obtain an approximate solution, an N term expansion of ϕ_1 and ϕ_2 is constructed. This results in a set of $2(N+1)$ equations that must be solved in conjunction with eight boundary conditions. To solve for the unknowns we utilize the tau method as developed and extensively applied to ordinary differential equations by Fox and Parker [8]. Application of this method to (11) for $n = 0, 1, \ldots, N-4$ in each layer, in combination with the eight boundary conditions results in a set of $2(N+1)$ simultaneous equations which constitute an algebraic eigenvalue problem of the form $(A - Bc)X = 0$. The eigenvalues (i.e., c's) are then determined using a general matrix eigenvalue solver, namely the QR algorithm. To test the effectiveness of the spectral Chebyshev tau method, we have carried out extensive numerical calculations. Some of the representative results are given in the following.

In Tables I and II we compare our numerically calculated eigenvalues with those obtained by Yih's longwave asymptotic approach. For $\alpha = 1.0 \times 10^{-5}$, as shown in

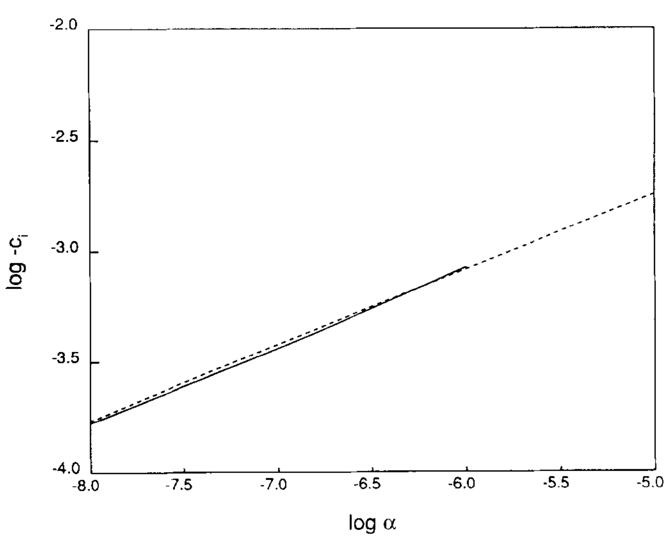

FIG. 1. Imaginary part of wave speed in the limit $\varepsilon \to \infty$, Re $= 1.0$, $r_2 = 1.0$, $m_2 = 2.0$, $\varepsilon = 1 \times 10^4$: ---- Yiantsios and Higgin's asymptotic results; ——— Numerical results.

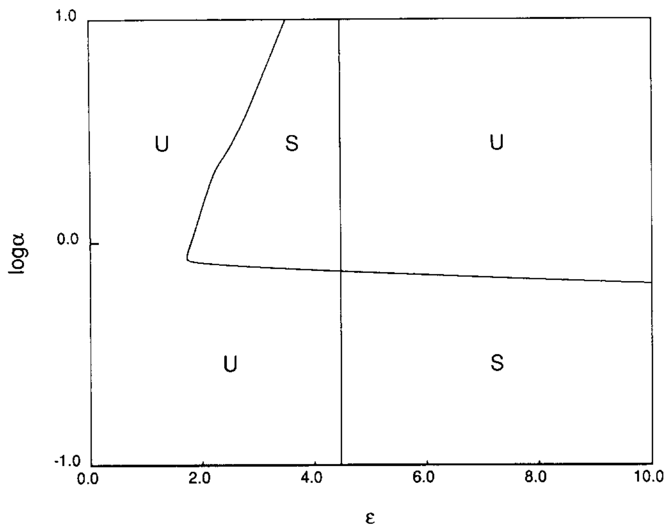

FIG. 2. Neutral stability contour for two superposed Newtonian fluids, $\mathrm{Re}=0.1$, $m_2=20$, $r_2=1.0$, $F=0$, $S=0$, $S=$ stable, $U=$ unstable.

Table I, the numerical results are indistinguishable from Yih's asymptotic results. However, as α is increased up to 1.0×10^{-2}, as depicted in Table II, our numerical and Yih's asymptotic results are slightly different. This is expected because the behavior of the interfacial mode can no longer be completely described by the method of longwave asymptotics due to an increase of the disturbance wavenumber (α). We have also compared our numerical results shown in Fig. 1 with the thin layer asymptotic results of Yiantsios and Higgins [3]. The numerical results are in complete agreement with the asymptotic results.

Unlike the compound matrix method, not only the spectral Chebyshev tau method can obtain all eigenvalues of the spectrum but also determines them accurately without a need for a good initial guess. Hence, this method enables us to determine the eigenvalue describing the linear stability of systems under consideration at any disturbance wavelengths in one step. A typical stability contour is shown in Fig. 2. In all our computations less than 30 terms (i.e., $N\leqslant 30$) were required to obtain convergent eigenvalues for $\alpha\leqslant 3.0$ with double-precision arithmetic. However, for $\alpha>3.0$ due to significant round-off errors, quadruple precision arithmetic had to be utilized. The neutral stability contours generated with this numerical technique are in complete agreement with results of Yiantsios and Higgins for $\alpha\leqslant 3.0$ while for larger α's some minor deviations are observed in the absence of interfacial tension.

4. TWO SUPERPOSED POWER LAW FLUIDS

Detail derivation of the stability governing equations for two superposed power law fluids has been carried out in an earlier publication [9] by us. Hence, no derivation of the governing equations will be presented in this section.

The constitutive equations used in this study is the power law model,

$$\mathbf{T}_k = m_k\left(\tfrac{1}{2}\mathrm{II}\;\dot{\gamma}_k\right)_{n_k-1/2}\dot{\gamma}_k, \tag{13}$$

where

$$\dot{\gamma}_k = (\nabla\mathbf{U}_k + \nabla\mathbf{U}_k^t), \qquad k=1,2.$$

The unperturbed velocity profiles for this model are

$$\begin{aligned}
U_1 &= \frac{n_1}{\mathrm{Re}\,L(n_1+1)}\left\{(\mathrm{Re}(Ly+b_1))^{(n_1+1)/n_1}\right.\\
&\quad\left. -(\mathrm{Re}(L+b_1))^{(n_1+1)/n_1}\right\}\\
U_2 &= \frac{n_2 h_2}{\mathrm{Re}\,L(n_2+1)}\left\{\left(\frac{\mathrm{Re}}{h_2}(Ly+b_1)\right)^{(n_2+1)/n_2}\right.\\
&\quad\left. -\left(\frac{\mathrm{Re}}{h_2}(b_1-\varepsilon L)\right)^{(n_2+1)/n_2}\right\},
\end{aligned} \tag{14}$$

where

$$h_2 = \left(\frac{U_0}{d_1}\right)^{n_2-n_1} R_m$$

$$\mathrm{Re} = \frac{\rho_1 U_0^{2-n_1} d_1^{n_1}}{m_1}$$

$$R_m = m_2/m_1$$

and L and b_1 are simply obtained from the continuity of normalized velocity at the interface by a Newton-Raphson procedure. The stability governing equations are

$$\begin{aligned}
&i\alpha r_k\left[(U_k-c)(\phi_k''-\alpha^2\phi_k)-\phi_k\frac{d\dot{\gamma}_k}{dy}\right]\\
&\quad=\phi_k''''\left(\frac{n_k h_k}{\mathrm{Re}}\dot{\gamma}_k^{n_k-1}\right)+\phi_k'''\left(\frac{2n_k h_k}{\mathrm{Re}}\right)\left[\frac{d}{dy}(\dot{\gamma}_k^{n_k-1})\right]\\
&\qquad+\phi_k''\left\{\left[\frac{2h_k\alpha^2(n_k-2)}{\mathrm{Re}}\right]\right.\\
&\qquad\left.\times(\dot{\gamma}_k^{n_k-1})+\left(\frac{n_k h_k}{\mathrm{Re}}\right)\left[\frac{d^2}{dy^2}(\dot{\gamma}_k^{n_k-1})\right]\right\}\\
&\qquad+\phi_k'\left\{\left[\frac{2h_k\alpha^2}{\mathrm{Re}}(n_k-2)\right]\left[\frac{d}{dy}(\dot{\gamma}_k^{n_k-1})\right]\right\}\\
&\qquad+\phi_k\left\{\left(\frac{h_k\alpha^4 n_k}{\mathrm{Re}}\right)(\dot{\gamma}_k^{n_k-1})\right.\\
&\qquad\left.+\left(\frac{n_k h_k\alpha^2}{\mathrm{Re}}\right)\left[\frac{d^2}{dy^2}(\dot{\gamma}_k^{n_k-1})\right]\right\}, \qquad k=1,2
\end{aligned} \tag{15}$$

the no slip boundary conditions and the continuity of velocities and stresses at the interface are

$$\phi_1 = \phi_1' = 0 \quad \text{at} \quad y = 1$$
$$\phi_2 = \phi_2' = 0 \quad \text{at} \quad y = -\varepsilon$$
$$\phi_1 = \phi_2 \quad \text{at} \quad y = 0$$
$$\phi_1' - \phi_2' = \phi_1(\dot{\gamma}_2(0) - \dot{\gamma}_1(0))/(c-1) \quad \text{at} \quad y = 0$$
$$n_1 h_1(\alpha^2\phi_1 + \phi_1'')\,\dot{\gamma}_1^{n_1-1} = n_2 h_2\{\alpha^2\phi_2 + \phi_2''\}\,\dot{\gamma}_2^{n_2-1} \quad \text{at} \quad y = 0$$
$$\begin{aligned} n_2 h_2 &\left\{(\alpha^2\phi_2' + \phi_2''')\,\dot{\gamma}_2^{n_2-1} + \left(\frac{d}{dy}\dot{\gamma}_2^{n_2-1}\right)(\alpha^2\phi_2 + \phi_2'')\right\} \\ &- n_1\left\{(\alpha^2\phi_1' + \phi_1''')\,\dot{\gamma}_1^{n_1-1} + \left(\frac{d}{dy}\dot{\gamma}_1^{n_1-1}\right)(\alpha^2\phi_1 + \phi_1'')\right\} \\ &+ 4\alpha^2\phi_1'\dot{\gamma}_1^{n_1-1} - 4h_2\alpha^2\phi_2'\dot{\gamma}_2^{n_2-1} \\ &+ \mathrm{Re}\{-i\alpha c\phi_1' + i\alpha(\phi_1' - \phi_1\dot{\gamma}_1)\} \\ &- r_2\,\mathrm{Re}\{-i\alpha c\phi_2' + i\alpha(\phi_2' - \phi_2\dot{\gamma}_2)\} \\ &= i\alpha\,\mathrm{Re}(F + \alpha^2 S)\,\phi/(c-1) \quad \text{at} \quad y = 0. \end{aligned} \tag{16}$$

Unlike the Newtonian case, the governing equations do not have polynomial coefficients; hence the pseudo-spectral Chebyshev method is employed. First the stability governing equations are transformed in the range of $-1 \leqslant y \leqslant 1$ and then approximations to ϕ_1 and ϕ_2 of the following type are assumed:

$$\begin{aligned} \phi_1 &= \sum_{n=1}^{N} a_n^{(1)} T_n(y) \\ \phi_2 &= \sum_{n=1}^{N} a_n^{(2)} T_n(y). \end{aligned} \tag{17}$$

Upon substitution of (17) into the governing equations and utilizing the WR conditions (6), we obtain the discretized governing equations

$$\begin{aligned} \sum_{n=0}^{N} &\left\{ \frac{p_k^1(y_j)}{24c_n} \sum_{\substack{p=n+4 \\ p\equiv n(\mathrm{mod}\,2)}}^{N} p[p^2(p^2-4)^2 - 3n^2p^4 + 3n^4p^2 - n^2(n^2-4)^2]\,a_p^{(k)} \right. \\ &+ \frac{8p2_k(y_j)}{c_n}[(n+1)(n+2)(n+3)\,a_{n+3}^{(k)} + 3(n+2)(n+3)(n+5)\,a_{n+5}^{(k)} + 6(n+3)(n+4)(n+7)\,a_{n+7}^{(k)} + \cdots] \\ &+ \frac{p3_k(y_j)}{c_n} \sum_{\substack{p=n+2 \\ p\equiv n(\mathrm{mod}\,2)}}^{N} p(p^2-n^2)\,a_p^{(k)} \\ &+ \frac{2p4_k(y_j)}{c_n} \sum_{\substack{p=n+1 \\ p+n\equiv 1(\mathrm{mod}\,2)}}^{N} pa_p^{(k)} + p5_k(y_j)\,a_n^{(k)} \\ &+ i\left[p7_k(y_j)\,a_n^{(k)} - \frac{p6_k(y_j)}{c_n} \sum_{\substack{p=n+2 \\ p\equiv n(\mathrm{mod}\,2)}}^{N} p(p^2-n^2)\,a_p^{(k)}\right] \\ &+ ic\left[\frac{4\alpha r_k}{(1+\varepsilon)^2 c_n} \sum_{\substack{p=n+2 \\ p\equiv n(\mathrm{mod}\,2)}}^{N} p(p^2-n^2)\,a_p^{(k)} \left. - \alpha^3 r_k a_n^{(k)}\right]\right\} T_n(y_j) = 0, \end{aligned} \tag{18}$$

where $p_k^1(y_j)$ to $p7_k(y_j)$ are constant parameters that are given in the Appendix. The boundary conditions in term of (17) are

$$\sum_{n=0}^{N} a_n^{(1)} = 0$$
$$\sum_{n=0}^{N} n^2 a_n^{(1)} = 0$$
$$\sum_{n=0}^{N} (-1)^n a_n^{(2)} = 0$$
$$\sum_{n=1}^{N} (-1)^{n-1} n^2 a_n^{(2)} = 0$$
$$\sum_{n=0}^{N} (a_n^{(1)} - a_n^{(2)})\,T_n(\varepsilon_1) = 0$$

$$\begin{aligned} \sum_{n=0}^{N} &\left\{\left[\frac{2BE1}{c_n} \sum_{\substack{p=n+1 \\ p+n\equiv 1(\mathrm{mod}\,2)}}^{N} pa_p^{(2)} - \frac{2BE2}{c_n} \times \sum_{\substack{p=n+1 \\ p+n\equiv 1(\mathrm{mod}\,2)}}^{N} pa_p^{(1)} + BE3a_n^{(1)} - BE4a_n^{(2)}\right]\right. \\ &\left. + \frac{2c}{c_n}\left[\sum_{\substack{p=n+1 \\ p+n\equiv 1(\mathrm{mod}\,2)}}^{N} pa_p^{(1)} - \sum_{\substack{p=n+1 \\ p+n\equiv 1(\mathrm{mod}\,2)}}^{N} pa_p^{(2)}\right]\right\} T_n(\varepsilon_1) = 0 \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{N} &\left\{[BE8a_n^{(1)} - BE9a_n^{(2)}] + \left[\frac{BE6}{c_n} \sum_{\substack{p=n+2 \\ p\equiv n(\mathrm{mod}\,2)}}^{N} p(p^2-n^2)\,a_p^{(1)} \right.\right. \\ &\left.\left. - \frac{BE7}{c_n} \sum_{\substack{p=n+2 \\ p\equiv n(\mathrm{mod}\,2)}}^{N} p(p^2-n^2)\,a_p^{(2)}\right]\right\} T_n(\varepsilon_1) = 0 \end{aligned}$$

$$\sum_{n=0}^{N} \left\{\left[BE12a_n^{(1)} + BE19a_n^{(2)} + \frac{2BE10}{c_n} \sum_{\substack{p=n+1 \\ p+n\equiv 1(\mathrm{mod}\,2)}}^{N} pa_p^{(1)} + \frac{2BE17}{c_n} \sum_{\substack{p=n+1 \\ p+n\equiv 1(\mathrm{mod}\,2)}}^{N} pa_p^{(2)}\right.\right.$$

$$+\frac{BE13}{c_n}\sum_{\substack{p=n+2\\ p\equiv n(\bmod 2)}}^{N} p(p^2-n^2)\,a_p^{(1)}$$
$$+\frac{BE20}{c_n}\sum_{\substack{p=n+2\\ p\equiv n(\bmod 2)}}^{N} p(p^2-n^2)\,a_p^{(2)}$$
$$+\frac{8BE11}{c_n}\,[(n+1)(n+2)(n+3)\,a_{n+3}^{(1)}$$
$$+3(n+2)(n+3)(n+5)\,a_{n+5}^{(1)}+\cdots]$$
$$+\frac{8BE18}{c_n}\,[(n+1)(n+2)(n+3)\,a_{n+3}^{(2)}$$
$$+3(n+2)(n+3)(n+5)\,a_{n+5}^{(2)}+\cdots]$$
$$+i\Bigg[BE16a_n^{(1)}+BE23a_n^{(2)}$$
$$+\frac{2BE15}{c_n}\sum_{\substack{p=n+1\\ p+n\equiv 1(\bmod 2)}}^{N} pa_p^{(1)}+\frac{2BE22}{c_n}\sum_{\substack{p=n+1\\ p+n\equiv 1(\bmod 2)}}^{N} pa_p^{(2)}\Bigg]$$
$$-ic\Bigg[-\frac{2BE21}{c_n}\sum_{\substack{p=n+1\\ p+n\equiv 1(\bmod 2)}}^{N} pa_p^{(2)}$$
$$+\frac{2BE14}{c_n}\sum_{\substack{p=n+1\\ p+n\equiv 1(\bmod 2)}}^{N} pa_p^{(1)}\Bigg]\Bigg\}\,T_n(\varepsilon_1)=0, \tag{19}$$

where $BE1$ to $BE23$ are constant parameters that are given in the Appendix.

Similar to the Newtonian case approximate solutions to ϕ_1 and ϕ_2 must be obtained. However, if N terms in (17) are used, $2(N-3)$ equations result from the discretization of stability governing equations which have to be solved in conjunction with eight boundary conditions. It has been shown earlier [10] that collocation approximation to Orr–Sommerfeld-type equations are less straightforward due to the double boundary conditions at the wall. Hence to avoid the overdeterminacy caused by these double boundary conditions a standard set of collocation points, i.e.,

$$y_j=\cos(\pi j/N) \tag{20}$$

have been proposed for $j=2, ..., N-2$, dropping the differential equation conditions at $j=1$ and $j=N-1$. This procedure achieves spectral accuracy but gives rise to a non-optimal order error. Herbert [11] has devised another collocation method, where he replaces the collocation points suggested by Eq. (20) with

$$y_j=\cos\left[\frac{\pi j}{N-4}\right], \qquad j=0, 1, ..., N-4, \tag{21}$$

to obtain spectral accuracy with an optimal order error.

We have generalized (21) to account for arbitrary depth ratios. The transformed version of this equation for each layer is

$$\begin{aligned} y_j&=(1-\varepsilon_1)\cos\left[\frac{\pi j}{2(N-4)}\right]+\varepsilon_1,\\ &\qquad 1\geqslant y\geqslant\varepsilon_1, j=0, 1, ..., N-4;\\ y_j&=(1+\varepsilon_1)\cos\left[\frac{\pi j}{2(N-4)}\right]+\varepsilon_1,\\ &\qquad \varepsilon_1\geqslant y\geqslant -1, j=N-4, N-3, ..., 2(N-4). \end{aligned} \tag{22}$$

Utilizing the above collocation points, the stability problem is reduced to a general eigenvalue problem and the QR algorithm is utilized to compute the eigenvalues.

Table III shows a typical comparison between our numerical results and asymptotic results of Khomami [9]. The numerical results are in excellent agreement with the asymptotics results. This method also enables us to determine the linear stability of this system at all disturbance wave lengths. A typical stability contour (i.e., for a power law fluid, $n_2=0.5$ and a Newtonian fluid $n_1=1.0$) is depicted in Fig. 3. This figure clearly indicates the dramatic effect of shear thinning viscosity on the stability contour. This issue is beyond the scope of this paper and is discussed elsewhere [9].

In order to compare the convergence rate of spectral and pseudospectral techniques, we have also considered the interfacial stability of two superposed Newtonian fluids with the pseudospectral algorithm. Both algorithms require at least 10 terms of the expansion to accurately estimate the longwave behavior. However, in the intermediate wave numbers (i.e., $\alpha\leqslant 3$) at least 14 and 18 terms were required to obtain convergent eigenvalues with the pseudospectral and spectral techniques, respectively. Hence, our results indicate that the pseudospectral algorithm is more efficient when applied to linear stability problems.

It is worth mentioning that both the spectral tau and pseudospectral method have been shown to give rise to spurious eigenvaues in addition to physical ones in the linear stability analysis of single phase incompressible plane shear flows [5]. In our studies, we also observed similar spurious modes which behave in a similar fashion to those obtained in single phase studies.

TABLE III

Interfacial Mode Eigenvalues for Two Superposed Power Law Fluids

n_1	n_2	R_m	c (Ref. [9])	c (Pseudospectral Method)
0.5	1.0	0.5	1.1867 + 1.6617 E-5	1.18664 + 1.66139 E-5
1.0	0.5	2.0	1.2385 + 7.9184 E-6	1.23844 + 7.91700 E-6
1.0	0.75	2.0	1.1350 + 3.8922 E-6	1.13498 + 3.89242 E-6
0.75	1.0	0.5	1.1236 + 8.5941 E-6	1.12355 + 8.59516 E-6
1.0	0.35	2.0	1.3129 + 1.1378 E-5	1.31285 + 1.13739 E-5
0.35	1.0	0.5	1.2170 + 2.1434 E-5	1.21702 + 2.14284 E-5

Note. $\alpha=1.0\times10^{-2}$, Re = 0.1, $r_2=1.0$, $F=0$, $S=0$, $\varepsilon=1.0$.

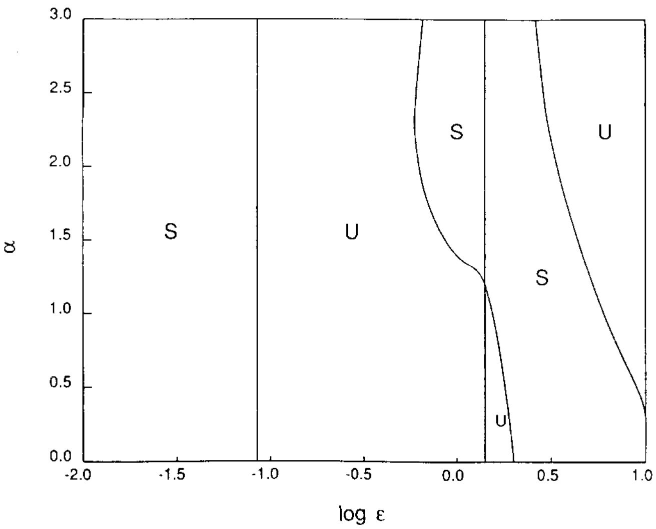

FIG. 3. Neutral stability diagram for two superposed power law fluids, $\mathrm{Re}=0.1$, $R_m=2.0$, $n_1=1.0$, $n_2=0.5$, $r_2=1.0$, $F=0$, $S=0$; $S=$ stable, $U=$ unstable.

CONCLUSIONS

Two spectral codes, namely, spectral Chebyshev tau and pseudospectral-Chebyshev, were developed to solve eigenvalue problems for equation sets valid over connected domains coupled through interfacial conditions. The Chebyshev tau method is applicable to equation sets with polynomial type coefficients while the pseudospectral technique is applicable to equation sets with arbitrary type coefficient. As we have shown in this paper both of these methods are capable of accurately solving eigenvalue problems with interfacial boundary conditions.

The pseudospectral technique outlined in this paper is superior to its spectral counterpart when applied to linear stability problems, due to its faster convergence and the ability to handle differential equations with arbitrary coefficients. However, the accuracy of the results obtained with this technique is highly dependent on the choice of collocation points. Hence, a proper selection of collocation points is crucial in successful application of this method.

APPENDIX

$$p1_k(y_j)=\frac{16}{(1+\varepsilon)^4}\left[\frac{n_k h_k}{\mathrm{Re}}\right][\dot\gamma_k(y_j)]^{n_k-1}$$

$$p2_k(y_j)=\frac{8}{(1+\varepsilon)^3}\left[\frac{2n_k h_k}{\mathrm{Re}}\right]\left[\frac{d\dot\gamma_k^{n_k-1}}{dy}(y_j)\right]$$

$$p3_k(y_j)=\frac{4}{(1+\varepsilon)^2}\left\{\left[\frac{2h_k\alpha^2(n_k-2)}{\mathrm{Re}}\right]\right.$$

$$\left.\times[\dot\gamma_k(y_j)]^{n_k-1}+\frac{n_k h_k}{\mathrm{Re}}\left[\frac{d^2\dot\gamma_k^{n_k-1}}{dy^2}(y_j)\right]\right\}$$

$$p4_k(y_j)=\frac{2}{(1+\varepsilon)}\left[\frac{2h_k\alpha^2(n_k-2)}{\mathrm{Re}}\frac{d\dot\gamma_k^{n_k-1}}{dy}(y_j)\right]$$

$$p5_k(y_j)=\left\{\frac{h_k\alpha^4 n_k}{\mathrm{Re}}[\dot\gamma_k(y_j)]^{n_k-1}\right.$$

$$\left.+\frac{n_k h_k\alpha^2}{\mathrm{Re}}\frac{d^2\dot\gamma_k^{n_k-1}}{dy^2}(y_j)\right\}$$

$$p6_k(y_j)=\frac{4U_k\alpha r_k}{(1+\varepsilon)^2}$$

$$p7_k(y_j)=r_k U_k\alpha^3+\alpha r_k\frac{d\dot\gamma_k}{dy}(y_j)$$

$$BE1=U_2(0)$$

$$BE2=U_1(0)$$

$$BE3=\frac{1+\varepsilon}{2}\frac{dU_1}{dy}(0)$$

$$BE4=\frac{1+\varepsilon}{2}\frac{dU_2}{dy}(0)$$

$$BE5=\frac{n_2 h_2}{n_1}\frac{\dot\gamma_2^{n_2-1}(0)}{\dot\gamma_1^{n_1-1}(0)}$$

$$BE6=\frac{4}{(1+\varepsilon)^2}$$

$$BE7=BE5\frac{4}{(1+\varepsilon)^2}$$

$$BE8=\alpha^2$$

$$BE9=\alpha^2\cdot BE5$$

$$BE10=\frac{2\alpha^2\dot\gamma_1^{n_1-1}(0)}{1+\varepsilon}(4-n_1)$$

$$BE11=\frac{-8n_1\dot\gamma_1^{n_1-1}(0)}{(1+\varepsilon)^3}$$

$$BE12=-n_1\alpha^2\frac{d\dot\gamma_1^{n_1-1}}{dy}(0)$$

$$BE13=\frac{-4n_1(d\dot\gamma_1^{n_1-1}/dy)(0)}{(1+\varepsilon)^2}$$

$$BE14=\frac{2\,\mathrm{Re}\,\alpha}{1+\varepsilon}$$

$$BE15=\frac{2\,\mathrm{Re}\,\alpha U_1(0)}{1+\varepsilon}$$

$$BE16=-\mathrm{Re}\,\alpha\dot\gamma_1(0)$$

$$BE17=\frac{2(n_2-4)\,\alpha^2 h_2\dot\gamma_2^{n_2-1}(0)}{(1+\varepsilon)}$$

$$BE18=\frac{8n_2 h_2\dot\gamma_2^{n_2-1}(0)}{(1+\varepsilon)^3}$$

$$BE19 = n_2 h_2 \alpha^2 \frac{d\dot{\gamma}_2^{n_2-1}}{dy}(0)$$

$$BE20 = \frac{4n_2 h_2}{(1+\varepsilon)^2} \frac{d\dot{\gamma}_2^{n_2-1}}{dy}(0)$$

$$BE21 = \frac{2r_2 \operatorname{Re} \alpha}{1+\varepsilon}$$

$$BE22 = \frac{-2r_2 \operatorname{Re} \alpha U_2(0)}{1+\varepsilon}$$

$$BE23 = \alpha r_2 \operatorname{Re} \dot{\gamma}_2(0).$$

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